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Research Article

An Adaptive Geodesic-Relaxed Extragradient Method for Equilibrium and Fixed-Point Problems in CAT(0) Spaces

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ABSTRACT

We propose an adaptive geodesic-relaxed extragradient method for finding a common solution of a pseudomonotone equilibrium problem and a fixed-point problem in a complete CAT(0) space. The method combines a two-point geodesic memory step, two regularized equilibrium subproblems, an adaptive stepsize rule, and a fixed-point correction. The terminology "geodesic-relaxed" is used deliberately: the memory point is a geodesic interpolation of two successive iterates and is not an extrapolation beyond the current iterate. The CAT(0) geometry yields the strong convexity of squared distance needed to derive the three-point inequalities underlying the extragradient descent estimate. The adaptive stepsizes are shown to remain bounded away from zero without requiring the Lipschitz-type constants in the implementation. Boundedness follows for quasi-nonexpansive fixed-point mappings. When the fixed-point mapping is a contraction, the whole sequence converges strongly to the unique common solution, and the equilibrium and fixed-point residuals vanish. For the broader nonexpansive case, a cluster-point result is stated under an explicit asymptotic-regularity and Δ-demiclosedness hypothesis. A reproducible one-dimensional experiment with a nontrivial contractive fixed-point map illustrates the theoretical behavior without claiming universal acceleration.

1. Introduction

Equilibrium problems provide a unified formulation for optimization, variational inequalities, game-theoretic models, and related nonlinear problems. Let C be a nonempty closed convex subset of a metric space and let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying $f(x, x) = 0$. The equilibrium problem is

$$\text{find } x^* \in C \quad \text{such that} \quad f(x^*, y) \geq 0, \qquad y \in C.$$

Its solution set is denoted by

$$EP(f, C) = \{x \in C : f(x, y) \geq 0 \text{ for every } y \in C\}.$$

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A common equilibrium and fixed-point problem seeks a point in

$$\Omega := EP(f, C) \cap Fix(T), \qquad Fix(T) := \{x \in C : Tx = x\}.$$

Extragradient-type methods are particularly useful for pseudomonotone equilibrium problems because they employ two regularized equilibrium subproblems and therefore can retain convergence properties beyond the monotone setting. Adaptive stepsize strategies are also attractive because the Lipschitz-type constants of a bifunction are rarely known accurately in applications.

In nonlinear metric geometry, vector inertial expressions such as $x_n + \theta_n(x_n - x_{n-1})$ are not intrinsically defined. A natural intrinsic alternative is to combine successive iterates along their geodesic. The point

$$w_n = (1 - \theta_n)x_n \oplus \theta_n x_{n-1}, \qquad 0 \le \theta_n \le 1,$$

lies on the geodesic segment joining x_n and x_{n-1}. It is therefore a *geodesic memory/relaxation step*, not an extrapolation beyond x_n. This distinction is important for both terminology and convergence analysis.

The present paper develops an adaptive geodesic-relaxed extragradient method in complete CAT(0) spaces. The restriction to CAT(0) geometry is deliberate: the squared distance is strongly geodesically convex, which permits the three-point inequalities required by the regularized equilibrium subproblems. The main contributions are:

- an intrinsic geodesic memory step coupled with an adaptive extragradient procedure;
- a stepsize rule that does not require the numerical values of the Lipschitz-type constants;
- a complete CAT(0) derivation of the fundamental three-point and descent inequalities;
- boundedness under quasi-nonexpansiveness, a Delta-cluster-point result under explicit regularity assumptions, and strong convergence under a contractive fixed-point constraint; and
- a reproducible numerical example with a nonidentity fixed-point map.

2. Preliminaries

Throughout, (X, d) is a complete CAT(0) space and $C \subset X$ is nonempty, closed, and convex. For $x, y \in X$ and $t \in [0, 1]$, the point

$$(1 - t)x \oplus ty$$

denotes the unique point at parameter t on the geodesic segment from x to y.

A basic CAT(0) inequality is

$$d^2\big((1 - t)x \oplus ty, z\big) \le (1 - t)d^2(x, z) + td^2(y, z) - t(1 - t)d^2(x, y). \tag{2.1}$$

Definition 2.1. *A bifunction $f : C \times C \to \mathbb{R}$ is pseudomonotone if*

$$f(x, y) \ge 0 \quad \Longrightarrow \quad f(y, x) \le 0$$

for all $x, y \in C$.

We impose the following assumptions on f.

Assumption 2.2. *The bifunction $f : C \times C \to \mathbb{R}$ satisfies:*

(A1) *$f(x, x) = 0$ for every $x \in C$;*
(A2) *f is pseudomonotone;*
(A3) *for every $x \in C$, the function $f(x, \cdot)$ is convex and lower semicontinuous;*
(A4) *there exist $c_1, c_2 > 0$ such that*

$$f(x, y) + f(y, z) \ge f(x, z) - c_1 d^2(x, y) - c_2 d^2(y, z)$$

for all $x, y, z \in C$;

(A5) *for every fixed* $y \in C$, *if* $x_k \xrightarrow{\Delta} x$, *then*

$$\limsup_{k\to\infty} f(x_k, y) \le f(x, y).$$

Assumption (A5) is an explicit Δ-upper-semicontinuity condition in the first variable. It replaces vague references to a "semicontinuity condition required in the limiting argument."

Definition 2.3. *A mapping* $T : C \to C$ *is* quasi-nonexpansive *if* $Fix(T) \neq \emptyset$ *and*

$$d(Tx, p) \le d(x, p), \qquad x \in C,\ p \in Fix(T).$$

It is a contraction if there exists $q \in [0, 1)$ *such that*

$$d(Tx, Ty) \le q\, d(x, y), \qquad x, y \in C.$$

Every contraction is nonexpansive and quasi-nonexpansive relative to its unique fixed point.

Lemma 2.4 (Three-point inequality). *Let* $a, b \in C$, $\lambda > 0$, *and let*

$$\bar{y} = \operatorname*{argmin}_{y\in C} \left\{ \lambda f(a, y) + \frac{1}{2} d^2(b, y) \right\}.$$

Then $\bar{y}$ *is unique and, for every* $y \in C$,

$$2\lambda \big[f(a, y) - f(a, \bar{y}) \big] \ge d^2(b, \bar{y}) + d^2(\bar{y}, y) - d^2(b, y). \tag{2.2}$$

Proof. By Assumption 2.2(A3) and the CAT(0) inequality (2.1), the function

$$\Phi(y) = \lambda f(a, y) + \frac{1}{2} d^2(b, y)$$

is strongly geodesically convex with modulus 1. Hence its minimizer is unique. For $t \in (0, 1)$ set $y_t = (1 - t)\bar{y} \oplus ty$. Minimality of $\bar{y}$ gives $\Phi(\bar{y}) \le \Phi(y_t)$. Convexity of $f(a, \cdot)$ and (2.1) yield

$$\Phi(y_t) \le (1 - t)\Phi(\bar{y}) + t\Phi(y) - \frac{1}{2} t(1 - t) d^2(\bar{y}, y).$$

After cancellation and division by t, let $t \downarrow 0$. Rearrangement gives (2.2).

Remark 2.5. *Lemma 2.4 is the precise geometric justification for the three-point estimates used below; no unproved Hilbert-space identity is imported into the metric setting.*

3. Adaptive Geodesic-Relaxed Extragradient Method

Let $u, x_0, x_1 \in C$, choose $\lambda_1 > 0$ and $\sigma \in (0, 1)$, and let $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\theta_n\}$ satisfy

$$\alpha_n \in (0, 1), \qquad \alpha_n \to 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty, \tag{3.1}$$

$$0 < \underline{\beta} \le \beta_n \le \overline{\beta} < 1, \tag{3.2}$$

$$\theta_n \in [0, 1], \qquad \frac{\theta_n d(x_n, x_{n-1})}{\alpha_n} \to 0. \tag{3.3}$$

Algorithm 1 Adaptive Geodesic-Relaxed Extragradient Method

Initialization: Choose $u, x_0, x_1 \in C$, $\lambda_1 > 0$, $\sigma \in (0,1)$, and sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\theta_n\}$ satisfying (3.1)–(3.3).

For $n = 1, 2, \ldots$**, perform the following steps:**
Step 1: Geodesic memory step

$$w_n = (1-\theta_n)x_n \oplus \theta_n x_{n-1}.$$

Step 2: First equilibrium step

$$y_n = \operatorname*{argmin}_{y\in C}\left\{\lambda_n f(w_n, y) + \frac{1}{2}d^2(w_n, y)\right\}.$$

Step 3: Extragradient step

$$z_n = \operatorname*{argmin}_{y\in C}\left\{\lambda_n f(y_n, y) + \frac{1}{2}d^2(w_n, y)\right\}.$$

Step 4: Adaptive stepsize

$$\Delta_n = f(w_n, z_n) - f(w_n, y_n) - f(y_n, z_n).$$

$$\lambda_{n+1} = \min\left\{\lambda_n, \frac{\sigma\left[d^2(w_n, y_n) + d^2(y_n, z_n)\right]}{2[\Delta_n]_+}\right\}, \tag{3.4}$$

where $[t]_+ = \max\{t, 0\}$; if $[\Delta_n]_+ = 0$, the second term is interpreted as $+\infty$.
Step 5: Fixed-point correction

$$v_n = (1-\beta_n)z_n \oplus \beta_n T z_n.$$

Step 6: Anchored update

$$x_{n+1} = \alpha_n u \oplus (1-\alpha_n)T v_n.$$

Set $n \leftarrow n+1$ **and repeat Steps 1–6.**

Lemma 3.1. *The sequence* $\{\lambda_n\}$ *generated by* (3.4) *is nonincreasing and*

$$\lambda_n \geq \underline{\lambda} := \min\left\{\lambda_1, \frac{\sigma}{2\max\{c_1, c_2\}}\right\} > 0.$$

Consequently, $\lambda_n \to \lambda_\infty > 0$ *and* $\lambda_n/\lambda_{n+1} \to 1$.

Proof. Assumption 2.2(A4) gives

$$\Delta_n \leq c_1 d^2(w_n, y_n) + c_2 d^2(y_n, z_n) \leq \max\{c_1, c_2\} R_n,$$

where

$$R_n := d^2(w_n, y_n) + d^2(y_n, z_n).$$

If $[\Delta_n]_+ > 0$, then

$$\frac{\sigma R_n}{2[\Delta_n]_+} \geq \frac{\sigma}{2\max\{c_1, c_2\}}.$$

If $[\Delta_n]_+ = 0$, then $\lambda_{n+1} = \lambda_n$. Induction yields the lower bound. Monotonicity follows directly from the minimum in (3.4). The remaining conclusions follow because a nonincreasing positive sequence converges to a positive limit. ∎

4. Convergence Analysis

Assume throughout this section that

$$\Omega := EP(f, C) \cap Fix(T) \neq \emptyset.$$

Lemma 4.1. *Let $p \in \Omega$. Then*

$$d^2(z_n, p) \leq d^2(w_n, p) - d^2(w_n, y_n) - d^2(y_n, z_n) + 2\lambda_n\big[f(w_n, z_n) - f(w_n, y_n) - f(y_n, z_n)\big]. \tag{4.1}$$

Consequently,

$$d^2(z_n, p) \leq d^2(w_n, p) - \left(1 - \sigma\frac{\lambda_n}{\lambda_{n+1}}\right) R_n. \tag{4.2}$$

In particular, there exist $N \in \mathbb{N}$ and $\gamma > 0$ such that

$$d^2(z_n, p) \leq d^2(w_n, p) - \gamma R_n, \qquad n \geq N. \tag{4.3}$$

Proof. Apply Lemma 2.4 to the first subproblem with $a = b = w_n$, $\bar{y} = y_n$, and $y = z_n$:

$$2\lambda_n[f(w_n, z_n) - f(w_n, y_n)] \geq d^2(w_n, y_n) + d^2(y_n, z_n) - d^2(w_n, z_n).$$

Apply it to the second subproblem with $a = y_n$, $b = w_n$, $\bar{y} = z_n$, and $y = p$:

$$2\lambda_n[f(y_n, p) - f(y_n, z_n)] \geq d^2(w_n, z_n) + d^2(z_n, p) - d^2(w_n, p).$$

Since $p \in EP(f, C)$, $f(p, y_n) \geq 0$. Pseudomonotonicity therefore gives $f(y_n, p) \leq 0$. Adding the two inequalities yields (4.1). The adaptive rule implies

$$2\lambda_{n+1}[\Delta_n]_+ \leq \sigma R_n.$$

Since $\Delta_n \leq [\Delta_n]_+$,

$$2\lambda_n\Delta_n \leq \sigma\frac{\lambda_n}{\lambda_{n+1}} R_n,$$

which gives (4.2). By Lemma 3.1, $\lambda_n/\lambda_{n+1} \to 1$; since $\sigma < 1$, an eventual positive constant γ exists. ∎
Define

$$e_n := \theta_n d(x_n, x_{n-1}).$$

Then $e_n/\alpha_n \to 0$, hence $e_n \to 0$.

Lemma 4.2. *Suppose T is quasi-nonexpansive. Then $\{x_n\}$ and all auxiliary sequences generated by the algorithm are bounded.*

Proof. For $p \in \Omega$,

$$d(w_n, p) \le d(x_n, p) + e_n.$$

For all sufficiently large n, Lemma 4.1 yields $d(z_n, p) \le d(w_n, p)$. Quasi-nonexpansiveness and geodesic convexity give

$$d(v_n, p) \le (1 - \beta_n)d(z_n, p) + \beta_n d(Tz_n, p) \le d(z_n, p)$$

and

$$d(Tv_n, p) \le d(v_n, p).$$

Hence,

$$d(x_{n+1}, p) \le (1 - \alpha_n)d(x_n, p) + \alpha_n d(u, p) + (1 - \alpha_n)e_n.$$

Because $e_n/\alpha_n \to 0$, there exists $M > 0$ such that $e_n \le M\alpha_n$ eventually. A standard induction then gives boundedness of $\{x_n\}$, and the remaining sequences are bounded by the preceding estimates. ∎

4.1. A Delta-cluster-point result for the nonexpansive case

For the general nonexpansive case, the following regularity hypotheses are stated explicitly rather than being hidden in the proof.

Assumption 4.3. *Assume that:*

(R1) $d(x_{n+1}, x_n) \to 0$;

(R2) $I - T$ *is* Δ*-demiclosed at zero: if* $s_k \xrightarrow{\Delta} s$ *and* $d(s_k, Ts_k) \to 0$, *then* $s \in Fix(T)$.

Proposition 4.4. *Let T be nonexpansive and suppose Assumption 4.3 holds. Then*

$$d(w_n, y_n) \to 0, \qquad d(y_n, z_n) \to 0, \qquad d(x_n, z_n) \to 0,$$

and

$$d(z_n, Tz_n) \to 0.$$

Every Delta-cluster point of $\{x_n\}$ belongs to Ω.

Proof. The proof follows the descent mechanism of Lemma 4.1. Assumption 4.3(R1), $e_n \to 0$, and boundedness imply that the asymptotic distance levels of x_n, w_n, z_n, and v_n coincide. Therefore

$$d^2(w_n, p) - d^2(z_n, p) \to 0.$$

Equation (4.3) gives $R_n \to 0$, so $d(w_n, y_n) \to 0$ and $d(y_n, z_n) \to 0$. Since $d(x_n, w_n) = e_n \to 0$, it follows that $d(x_n, z_n) \to 0$.

The uniform convexity of CAT(0) spaces, together with $\underline{\beta} \le \beta_n \le \overline{\beta} < 1$, yields $d(z_n, Tz_n) \to 0$ from the fixed-point averaging step. Let $x_{n_k} \xrightarrow{\Delta} q$. Then $z_{n_k} \xrightarrow{\Delta} q$, and (R2) gives $q \in Fix(T)$.

To identify the equilibrium component, use the first subproblem and Lemma 2.4: for every fixed $y \in C$,

$$2\lambda_n[f(w_n, y) - f(w_n, y_n)] \ge d^2(w_n, y_n) + d^2(y_n, y) - d^2(w_n, y).$$

Along the Δ-convergent subsequence, the residual limits, the lower bound on λ_n, Assumption 2.2(A3), and the Delta-upper-semicontinuity in (A5) give $f(q, y) \ge 0$ for every $y \in C$. Thus $q \in EP(f, C)$ and therefore $q \in \Omega$. ∎

4.2. Strong convergence under a contractive fixed-point constraint

The next result provides a complete strong-convergence statement without using a sunny nonexpansive retraction or an unspecified variational selection term.

Theorem 4.5. *Suppose $T : C \to C$ is a contraction with constant $q \in [0, 1)$ and $\Omega \neq \emptyset$. Then $\Omega = \{p\}$ is a singleton and the sequence generated by the adaptive geodesic-relaxed extragradient method converges strongly to p:*

$$x_n \to p.$$

Moreover,

$$d(w_n, y_n) \to 0, \qquad d(y_n, z_n) \to 0, \qquad d(z_n, Tz_n) \to 0.$$

Proof. A contraction has a unique fixed point; because $\Omega \neq \emptyset$, that fixed point p also belongs to $EP(f, C)$.

Using geodesic convexity and contractivity,

$$d(v_n, p) \leq (1 - \beta_n)d(z_n, p) + \beta_n d(Tz_n, p) \leq d(z_n, p),$$

and

$$d(Tv_n, p) \leq q\, d(v_n, p) \leq q\, d(z_n, p).$$

For all sufficiently large n, Lemma 4.1 yields $d(z_n, p) \leq d(w_n, p) \leq d(x_n, p) + e_n$. Thus

$$d(x_{n+1}, p) \leq \alpha_n d(u, p) + (1 - \alpha_n)q\,[d(x_n, p) + e_n].$$

Let $L = \limsup_{n\to\infty} d(x_n, p)$. Since $\alpha_n \to 0$ and $e_n \to 0$,

$$L \leq qL.$$

As $0 \leq q < 1$, we obtain $L = 0$, proving $x_n \to p$.

Then $w_n \to p$ and, by the descent inequality, $z_n \to p$. From (4.3),

$$0 \leq \gamma R_n \leq d^2(w_n, p) - d^2(z_n, p) \leq d^2(w_n, p) \to 0,$$

so $R_n \to 0$, proving the two equilibrium residual limits. Finally,

$$d(z_n, Tz_n) \leq d(z_n, p) + d(Tz_n, p) \leq (1 + q)d(z_n, p) \to 0. \qquad \blacksquare$$

Corollary 4.6. *If $\theta_n = 0$ for all n, the method reduces to the corresponding adaptive non-memory extragradient scheme, and Theorem 4.5 remains valid.*

5. Numerical Illustration

This section gives a fully specified reproducible example. Its purpose is to illustrate the convergence mechanism; it is not presented as evidence of universal acceleration.

Let

$$C = [0, 1] \subset \mathbb{R}, \qquad a = \frac{1}{2},$$

and define

$$f(x, y) = (x - a)(y - x).$$

The associated equilibrium problem is the scalar variational inequality

$$(x^* - a)(y - x^*) \geq 0, \qquad y \in [0, 1],$$

whose unique solution is $x^* = a = 1/2$.

Choose the nontrivial contractive mapping

$$T(x) = 0.9x + 0.05.$$

Then T has contraction constant $q = 0.9$ and unique fixed point $1/2$. Hence

$$\Omega = EP(f, C) \cap Fix(T) = \left\{ \frac{1}{2} \right\}.$$

For this example the two equilibrium subproblems have the explicit forms

$$y_n = P_{[0,1]}\big(w_n - \lambda_n(w_n - a)\big)$$

and

$$z_n = P_{[0,1]}\big(w_n - \lambda_n(y_n - a)\big).$$

Thus the experiment requires no inner optimization solver.

For the numerical experiment, we choose

$$x_0 = x_1 = 0.9, \qquad u = 0.2, \qquad \alpha_n = \frac{1}{n+1}, \qquad \beta_n = 0.5,$$

and define

$$\varepsilon_n = \frac{1}{(n+1)^2}, \qquad \theta_n = \min\left\{ 0.30, \frac{\varepsilon_n}{|x_n - x_{n-1}| + \varepsilon_n} \right\}.$$

The initial stepsize and adaptive parameter are set to

$$\lambda_1 = 1, \qquad \sigma = 0.90.$$

These choices satisfy the control conditions imposed in (3.1)–(3.3). In particular, the definition of θ_n ensures that the contribution of the geodesic memory term becomes sufficiently small relative to α_n as the iteration proceeds.

Since the unique common solution is

$$x^* = \frac{1}{2},$$

the numerical error at iteration n is measured by

$$E_n = |x_n - x^*|.$$

A direct implementation of the proposed method with the above parameters produces the numerical results reported in Table 1.

As shown in Table 1, the iterates approach the unique common solution $x^* = 0.5$. The error is reduced from 4.0×10^{-1} at the initial iterate to approximately 2.72×10^{-2} after 50 iterations. The adaptive rule reduces the initial stepsize from $\lambda_1 = 1$ to 0.9, after which the stepsize remains unchanged over the reported iterations. These observations are consistent with the convergence behavior predicted by Theorem 4.5.

To visualize the convergence more clearly, Figure 1 displays the error E_n on a semilogarithmic scale.

Table 1
Numerical convergence of the proposed method.

Iteration n	x_n	Error $E_n = \|x_n - x^*\|$	Stepsize λ_n
0	0.900000	0.400000	1.0000
5	0.383879	0.116121	0.9000
10	0.393359	0.106641	0.9000
20	0.433327	0.066673	0.9000
30	0.454597	0.045403	0.9000
40	0.465980	0.034020	0.9000
50	0.472841	0.027159	0.9000

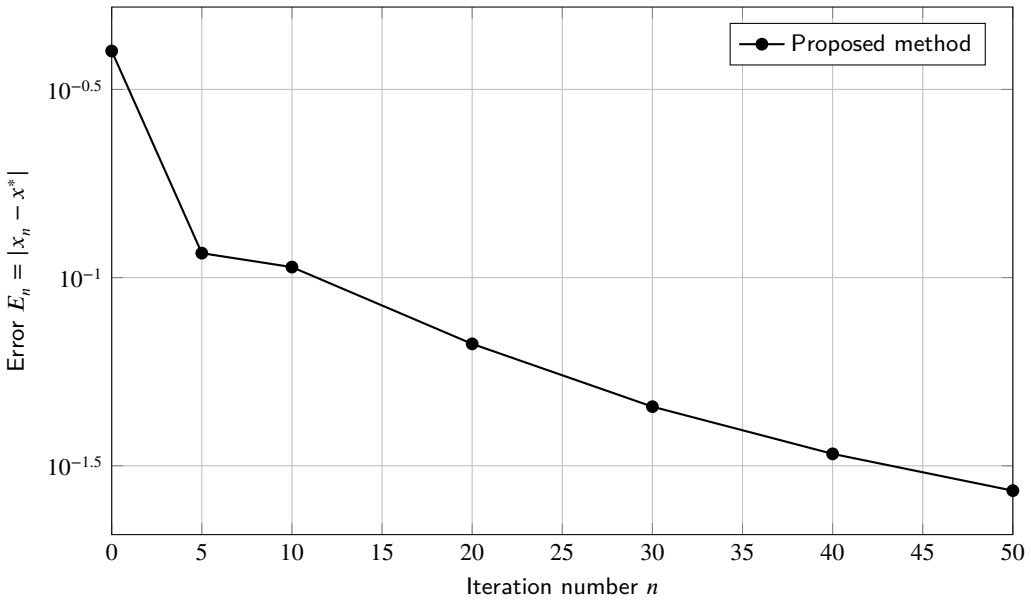


Figure 1: Convergence history of the proposed method measured by the error $E_n = |x_n - x^*|$.

Figure 1 provides a complementary view of the numerical behavior. The error exhibits an overall decreasing trend, although the decay is not uniformly geometric over the reported iteration range. The convergence becomes more gradual at later iterations, which is consistent with the diminishing anchored parameter $\alpha_n = 1/(n+1)$ and the choice $u = 0.2 \neq x^*$. Accordingly, this experiment should be interpreted as a numerical illustration of the convergence mechanism established in Theorem 4.5, rather than as evidence of a universal linear convergence rate or acceleration property. In particular, the experiment does not imply that the geodesic memory step improves the convergence rate for every problem or every parameter regime.

6. Discussion and Conclusion

We introduced an adaptive geodesic-relaxed extragradient method for a common pseudomonotone equilibrium and fixed-point problem in a complete CAT(0) space. The revision makes three structural points explicit.

First, the two-point point

$$w_n = (1 - \theta_n)x_n \oplus \theta_n x_{n-1}$$

is a geodesic interpolation and is therefore described as a memory/relaxation step rather than as inertial extrapolation. Second, the CAT(0) strong convexity of squared distance supplies the exact three-point inequality used in the extragradient analysis. Third, strong convergence is stated under a transparent contractive fixed-point assumption,

avoiding an unproved extension of CAT(0) demiclosedness results and avoiding an unspecified sunny nonexpansive retraction.

The adaptive stepsize remains bounded away from zero without requiring the Lipschitz-type constants in the implementation. Under quasi-nonexpansiveness the generated sequences are bounded. In the nonexpansive case, Delta-cluster points are common solutions under explicit asymptotic regularity and demiclosedness assumptions. Under a contractive fixed-point constraint, the whole sequence converges strongly and all principal residuals vanish.

Future work may investigate genuinely extrapolative geodesic constructions under suitable geodesic-extension assumptions, weaker fixed-point conditions, and adaptive rules that can be shown to improve complexity or convergence rates in identifiable problem classes.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Author Contributions

Sunisa Saiuparad: Conceptualization, methodology, investigation, and writing of the original draft. Kanikar Muangchoo: Conceptualization, methodology, validation, supervision, and writing–review and editing. Sukjit Tangcharoen: Formal analysis and validation. Pennapa Suwanbamrung: Investigation and numerical verification. Chanwit Prabpayak: Formal analysis and manuscript review. Pannika Mee-On: Validation and manuscript review. All authors read and approved the final manuscript.

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