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Research Article

# Golden-Ratio Resolvent Algorithm for Semiconvex Optimization in CAT(1) Spaces

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ABSTRACT

We introduce a Golden-Ratio Resolvent Algorithm (GRRA) for minimizing proper lower semicontinuous semiconvex functions on admissible regions of complete CAT(1) spaces. The method first evaluates the Moreau-Yosida resolvent and then performs a geodesic relaxation with the fixed parameter given by the reciprocal of the golden ratio. To avoid using a curvature estimate outside its valid range, the analysis is formulated on a common geodesically convex region on which a uniform local strong-convexity inequality for the squared distance is explicitly assumed. For semiconvex objectives, we do not identify resolvent fixed points with global minimizers; instead, the minimizer set is treated directly and the quasi-nonexpansiveness and $\Delta$-demiclosedness properties needed for global convergence are stated as explicit assumptions. Under these hypotheses, we establish a Fejér-type descent inequality, residual summability, asymptotic regularity, and $\Delta$-convergence to a minimizer. Under an additional contractivity assumption on the resolvent, strong convergence with an explicit linear rate follows. We further study stability under Mosco perturbations, including convergence of every fixed finite GRRA trajectory and a quantitative perturbation estimate under a local Lipschitz condition for geodesic interpolation. A spherical quadratic example gives an exact contraction factor, illustrates Mosco stability, and compares the golden-ratio relaxation with two fixed-parameter Krasnosel'skiĭ-Mann-type resolvent relaxations. The comparison is intended as an illustration and does not assert universal optimality of the golden-ratio parameter.

## 1. Introduction

Optimization in nonlinear metric spaces has received increasing attention because many problems in geometry, nonlinear analysis, variational theory, and optimization on manifolds are naturally formulated beyond linear spaces. A fundamental framework is provided by geodesic metric spaces with curvature bounded above in the sense of Alexandrov. In particular, CAT(0) spaces support extensive extensions of convex analysis, proximal algorithms, and variational methods from Hilbert spaces to nonlinear settings [2, 3, 7]. The positively curved CAT(1) setting is substantially more delicate because the convexity properties of the squared distance are only local.

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Semiconvex functions form a natural class of objectives in positively curved spaces. They allow a controlled departure from geodesic convexity while retaining enough variational structure for Moreau-Yosida regularization and gradient-flow constructions. Ohta and Pálfia developed a theory of gradient flows and Trotter-Kato approximation for semiconvex functions on CAT(1) spaces [5], while Gál and Pálfia studied convergence and stability of semiconvex functions in this geometry [6].

For a proper lower semicontinuous function $f$ defined on a suitable geodesically convex region $C$, the quadratic metric resolvent is

$$J_\lambda^f(x) := \operatorname{argmin}_{y\in C}\left\{ f(y) + \frac{1}{2\lambda} d^2(x,y)\right\}, \qquad \lambda > 0.$$

This is the intrinsic counterpart of the proximal mapping and requires no ambient vector-space structure. Resolvent methods are central in convex optimization and minimizing-movement schemes [2, 5, 9].

Variational convergence is equally important when the objective changes under perturbation. Mosco convergence combines a weak liminf inequality with strong recovery sequences and plays a central role in the stability of minimizers, envelopes, and resolvents [1, 4, 8]. In the CAT(1) setting, appropriate uniform semiconvexity and admissibility assumptions are required for analogous stability results [6].

Golden-ratio ideas have also been used in iterative optimization. Malitsky introduced golden-ratio algorithms for variational inequalities [10]; related primal-dual and nonlinear metric extensions were developed in [11, 12]. Let

$$\phi = \frac{1+\sqrt{5}}{2}, \qquad \vartheta = \frac{1}{\phi} = \phi - 1,$$

so that $1 - \vartheta = \vartheta^2$. In the present paper we prescribe this value as a fixed geodesic relaxation parameter in a resolvent iteration.

The proposed GRRA is

$$z_n = J_{\lambda_n}^f(x_n), \qquad x_{n+1} = (1-\vartheta)x_n \oplus \vartheta z_n, \qquad \vartheta = \frac{1}{\phi}. \tag{1.1}$$

The method is entirely intrinsic. The convergence mechanism, however, does not rely on an unsupported global CAT(1) convexity formula. Instead, we work on one common admissible region on which the precise local strong-convexity estimate used in the proof is stated explicitly.

A second point is essential for semiconvex objectives. When negative semiconvexity parameters are allowed, a fixed point of a quadratic resolvent need not be a global minimizer. Accordingly, we do not use the identity $Fix(J_\lambda^f) = \operatorname{argmin} f$ in the nonconvex semiconvex case. The global minimizer set is treated directly, and the quasi-nonexpansiveness and $\Delta$-demiclosedness properties needed for convergence to a minimizer are imposed explicitly.

The main contributions are as follows:

- We formulate an intrinsic golden-ratio relaxed resolvent iteration on admissible regions of complete CAT(1) spaces.
- Under an explicit local strong-convexity property for the squared distance and quasi-nonexpansiveness of the resolvent with respect to the minimizer set, we prove a Fejér-type descent inequality, residual summability, and asymptotic regularity.
- Under an additional $\Delta$-demiclosedness condition, we prove $\Delta$-convergence to a global minimizer. Under a further contractive resolvent assumption, strong convergence with an explicit linear rate is obtained.
- We establish finite-trajectory stability under Mosco perturbations. A quantitative perturbation recurrence is obtained under an explicit local Lipschitz estimate for geodesic interpolation.
- We give a spherical quadratic example with an exact resolvent and an exact scalar convergence factor. The numerical comparison is deliberately limited to the tested relaxation parameters and does not claim universal superiority of the golden-ratio choice.

The remainder of the paper is organized as follows. Section 2 reviews CAT(1) geometry, semiconvexity, Moreau-Yosida resolvents, Δ-convergence, and Mosco convergence. Section 3 introduces GRRA and establishes its fundamental descent properties. Section 4 presents the Δ-convergence and strong-convergence analysis. Section 5 studies stability under Mosco perturbations. Section 6 presents the spherical quadratic example and numerical comparisons. Section 7 concludes the paper.

## 2. Preliminaries

Throughout, $(X, d)$ denotes a complete CAT(1) space. To make the curvature dependence precise, we work on a nonempty closed geodesically convex set $C \subset X$ satisfying the following standing geometric condition.

**Assumption 2.1 (Admissible local convexity).** *There exists a constant $\sigma_C > 0$ such that, for every $x, y, p \in C$ and every $t \in [0, 1]$,*

$$d^2(p, (1-t)x \oplus ty) \le (1-t)d^2(p, x) + td^2(p, y) - \sigma_C t(1-t)d^2(x, y). \tag{2.1}$$

*All iterates, resolvent points, minimizers, and perturbations considered below are assumed to remain in $C$.*

**Remark 2.2.** *Assumption 2.1 is the exact geometric property required by the proofs. In CAT(1) geometry such an estimate is available on sufficiently small admissible regions. A convenient way to ensure the required locality is to work in a common geodesically convex region whose diameter is strictly below $\pi/2$ and to use the corresponding curvature-dependent lower bound supplied by the local convexity theorem; see [5, 6]. We intentionally use the symbol $\sigma_C$ rather than assigning a universal formula to an arbitrary ball of radius below $\pi/2$.*

If $d(x, y) < \pi$, the points $x$ and $y$ are joined by a unique geodesic segment, and

$$(1-t)x \oplus ty, \qquad t \in [0, 1],$$

denotes the point at geodesic parameter $t$. Thus

$$d(x, (1-t)x \oplus ty) = t\, d(x, y), \quad d((1-t)x \oplus ty, y) = (1-t)d(x, y).$$

### 2.1. Semiconvex functions

A proper function $f : X \to (-\infty, +\infty]$ is called $\alpha$-semiconvex on $C$ if, for every $x, y \in C$ and $t \in [0, 1]$,

$$f((1-t)x \oplus ty) \le (1-t)f(x) + tf(y) - \frac{\alpha}{2}t(1-t)d^2(x, y). \tag{2.2}$$

The cases $\alpha = 0$ and $\alpha > 0$ correspond to geodesic convexity and strong geodesic convexity, respectively. A negative $\alpha$ permits controlled nonconvexity.

### 2.2. Moreau–Yosida resolvent

For $\lambda > 0$ and $x \in C$, define

$$\Phi_{\lambda,x}(y) = f(y) + \frac{1}{2\lambda}d^2(x, y), \qquad y \in C.$$

The Moreau-Yosida envelope and quadratic resolvent are

$$f_\lambda(x) = \inf_{y \in C}\left\{ f(y) + \frac{1}{2\lambda}d^2(x, y)\right\}, \tag{2.3}$$

$$J_\lambda^f(x) = \operatorname*{argmin}_{y \in C}\left\{ f(y) + \frac{1}{2\lambda}d^2(x, y)\right\}. \tag{2.4}$$

By (2.2) and (2.1), $\Phi_{\lambda,x}$ has strong convexity modulus

$$m_\lambda = \alpha + \frac{\sigma_C}{\lambda}. \tag{2.5}$$

**Assumption 2.3 (Resolvent existence and admissibility).** *There are constants* $0 < \underline{\lambda} \le \overline{\lambda} < \infty$ *such that, for every* $\lambda \in [\underline{\lambda}, \overline{\lambda}]$,

$$\alpha + \frac{\sigma_C}{\lambda} > 0,$$

*and, for every* $x \in C$, *the functional* $\Phi_{\lambda,x}$ *attains its minimum in* $C$.

Under Assumption 2.3, the strong convexity of $\Phi_{\lambda,x}$ implies that $J_\lambda^f(x)$ is single valued.

Let $p = J_\lambda^f(x)$. The minimizing property gives

$$f(p) + \frac{1}{2\lambda} d^2(x,p) \le f(y) + \frac{1}{2\lambda} d^2(x,y), \qquad y \in C. \tag{2.6}$$

Moreover, strong convexity of $\Phi_{\lambda,x}$ yields

$$f(p) + \frac{1}{2\lambda} d^2(x,p) + \frac{m_\lambda}{2} d^2(p,y) \le f(y) + \frac{1}{2\lambda} d^2(x,y), \qquad y \in C. \tag{2.7}$$

Define the global minimizer set

$$\Omega := \operatorname*{argmin}_{y \in C} f(y).$$

**Proposition 2.4.** *If* $p \in \Omega$, *then* $J_\lambda^f(p) = p$ *for every admissible* $\lambda$.

**Proof.** For every $y \in C$,

$$f(y) + \frac{1}{2\lambda} d^2(p,y) \ge f(p),$$

with equality at $y = p$. Since the regularized functional has a unique minimizer, $J_\lambda^f(p) = p$.

**Remark 2.5 (Fixed points versus global minimizers).** *For a general semiconvex function with negative semiconvexity parameter, the converse of Proposition 2.4 need not hold. Thus, in general,*

$$\Omega \subseteq Fix(J_\lambda^f),$$

*but equality is not assumed. This distinction is essential in the nonconvex semiconvex setting. The convergence theorems below therefore target* $\Omega$ *directly through explicit assumptions.*

### 2.3. Δ-convergence and Mosco convergence

For a bounded sequence $\{x_n\} \subset C$ and $x \in C$, define

$$r(x, \{x_n\}) := \limsup_{n \to \infty} d(x, x_n).$$

The asymptotic center is the set of minimizers of this asymptotic radius. A sequence $\{x_n\}$ is said to Δ-converge to $x$, written $x_n \xrightarrow{\Delta} x$, if $x$ is the unique asymptotic center of every subsequence. On the admissible uniformly convex regions considered here, bounded sequences have Δ-convergent subsequences; see [5, 6].

A sequence of proper functions $f_m : X \to (-\infty, +\infty]$ is said to Mosco converge to $f$, written $f_m \xrightarrow{M} f$, if

1. whenever $x_m \xrightarrow{\Delta} x$,
$$f(x) \le \liminf_{m \to \infty} f_m(x_m);$$
2. for every $x \in X$, there exists $y_m \to x$ such that
$$f(x) \ge \limsup_{m \to \infty} f_m(y_m).$$

Under appropriate uniform semiconvexity, admissibility, and compactness or equi-coercivity assumptions, Mosco convergence implies convergence of the corresponding resolvents in the CAT(1) framework [6]. In Section 5 we state exactly the resolvent-stability property used by the algorithmic argument.

## 3. Main Results

Assume throughout this section that $\Omega \neq \varnothing$ and that Assumptions 2.1 and 2.3 hold.

### 3.1. Golden-Ratio Resolvent Algorithm

Let

$$\phi = \frac{1+\sqrt{5}}{2}, \qquad \vartheta = \frac{1}{\phi}.$$

Choose $x_0 \in C$ and a sequence $\{\lambda_n\} \subset [\underline{\lambda}, \overline{\lambda}]$. Define

$$z_n = J^f_{\lambda_n}(x_n), \qquad x_{n+1} = (1-\vartheta)x_n \oplus \vartheta z_n, \qquad n \geq 0. \tag{3.1}$$

**Algorithm 1** Golden-Ratio Resolvent Algorithm (GRRA)

Choose $x_0 \in C$ and $\lambda_n \in [\underline{\lambda}, \overline{\lambda}]$.
Set $\phi = (1+\sqrt{5})/2$ and $\vartheta = 1/\phi$.
**for** $n = 0, 1, 2, \ldots$ **do**
  $z_n \leftarrow J^f_{\lambda_n}(x_n)$.
  $x_{n+1} \leftarrow (1-\vartheta)x_n \oplus \vartheta z_n$.
**end for**

For numerical implementation one may terminate when $d(x_n, z_n)$ is below a prescribed tolerance. In the general semiconvex case this residual is not, by itself, a certificate of global optimality; convergence to a global minimizer is guaranteed only under the assumptions stated below.

**Assumption 3.1 (Quasi-nonexpansiveness with respect to minimizers).** *For every admissible $\lambda$, every $x \in C$, and every $p \in \Omega$,*

$$d(J^f_\lambda x, p) \leq d(x, p). \tag{3.2}$$

**Remark 3.2.** *Assumption 3.1 is not claimed to follow automatically from semiconvexity. It is the metric fixed-set property required for the Fejér estimate below. In convex or more structured settings it may follow from stronger resolvent inequalities, while in the general semiconvex CAT(1) setting it must be verified or imposed separately.*

### 3.2. Fundamental descent inequality

**Lemma 3.3.** *Suppose Assumption 3.1 holds. Then, for every $p \in \Omega$,*

$$d^2(x_{n+1}, p) \leq d^2(x_n, p) - \sigma_C \vartheta(1-\vartheta) d^2(x_n, z_n). \tag{3.3}$$

*Consequently, $\{x_n\}$ is Fejér monotone with respect to $\Omega$.*

**Proof.** By (3.1) and Assumption 2.1,

$$d^2(x_{n+1}, p) \leq (1-\vartheta)d^2(x_n, p) + \vartheta d^2(z_n, p) - \sigma_C \vartheta(1-\vartheta) d^2(x_n, z_n).$$

By Assumption 3.1, $d(z_n, p) \leq d(x_n, p)$. Substitution gives (3.3).

**Corollary 3.4.** *Under the assumptions of Lemma 3.3,*

$$\sum_{n=0}^{\infty} d^2(x_n, z_n) < \infty. \tag{3.4}$$

*Hence*

$$d(x_n, z_n) \to 0 \qquad \text{and} \qquad d(x_{n+1}, x_n) \to 0.$$

**Proof.** Fix $p \in \Omega$ and sum (3.3) from $n = 0$ to $N$:

$$\sigma_C \vartheta(1-\vartheta) \sum_{n=0}^{N} d^2(x_n, z_n) \le d^2(x_0, p).$$

Letting $N \to \infty$ proves (3.4). Since $x_{n+1}$ lies on the geodesic from $x_n$ to $z_n$,

$$d(x_{n+1}, x_n) = \vartheta d(x_n, z_n) \to 0.$$

The resolvent step also satisfies

$$f(z_n) + \frac{1}{2\lambda_n} d^2(x_n, z_n) \le f(x_n). \tag{3.5}$$

Equation (3.5) concerns the resolvent point $z_n$. For a general semiconvex objective, monotonicity of the sequence $\{f(x_n)\}$ is not asserted.

## 4. Convergence Analysis

### 4.1. Δ-convergence

**Assumption 4.1 (Δ-demiclosedness into the minimizer set).** *Let $\{u_k\} \subset C$ be bounded and $\lambda_k \in [\underline{\lambda}, \overline{\lambda}]$. If*

$$u_k \xrightarrow{\Delta} u \qquad \text{and} \qquad d(u_k, J^f_{\lambda_k} u_k) \to 0,$$

*then $u \in \Omega$.*

**Theorem 4.2.** *Suppose $\Omega \neq \varnothing$ and Assumptions 2.1, 2.3, 3.1, and 4.1 hold. Then the sequence generated by GRRA Δ-converges to a point $x^* \in \Omega$.*

**Proof.** By Lemma 3.3, for every $p \in \Omega$ the sequence $\{d(x_n, p)\}$ is nonincreasing and therefore convergent. Hence $\{x_n\}$ is bounded. By the admissible uniform convexity of $C$, every bounded sequence has a Δ-convergent subsequence. Let $x_{n_k} \xrightarrow{\Delta} \bar{x}$.

Corollary 3.4 gives

$$d(x_{n_k}, J^f_{\lambda_{n_k}} x_{n_k}) = d(x_{n_k}, z_{n_k}) \to 0.$$

Assumption 4.1 implies $\bar{x} \in \Omega$.

Thus every Δ-cluster point belongs to $\Omega$, while $d(x_n, p)$ has a limit for every $p \in \Omega$. The standard asymptotic-center argument for Fejér sequences in the present admissible CAT(1) region yields uniqueness of the Δ-cluster point. Therefore $x_n \xrightarrow{\Delta} x^*$ for some $x^* \in \Omega$.

### 4.2. Strong and linear convergence

**Assumption 4.3 (Contractive resolvent).** *The minimizer is unique, $\Omega = \{p\}$, and there exists $q \in [0, 1)$ such that, for every admissible $\lambda$ and every $x \in C$,*

$$d(J^f_\lambda x, p) \le q\, d(x, p). \tag{4.1}$$

**Theorem 4.4.** *Under Assumptions 2.1 and 4.3, the GRRA sequence converges strongly to $p$. More precisely,*

$$d(x_n, p) \le \rho^n d(x_0, p), \qquad \rho := \sqrt{1 - \vartheta(1-q^2)} < 1. \tag{4.2}$$

**Proof.** From Assumption 2.1,

$$d^2(x_{n+1}, p) \le (1-\vartheta) d^2(x_n, p) + \vartheta d^2(z_n, p)$$

$$
\begin{aligned}
&- \sigma_C \vartheta(1-\vartheta) d^2(x_n, z_n) \\
&\leq \left[(1-\vartheta) + \vartheta q^2\right] d^2(x_n, p).
\end{aligned}
$$

Since

$$(1-\vartheta) + \vartheta q^2 = 1 - \vartheta(1-q^2) < 1,$$

iteration and taking square roots yield (4.2).

**Remark 4.5.** *The rate in Theorem 4.4 is a general metric bound obtained only from the squared-distance convexity estimate. In special examples, including the spherical quadratic problem in Section 6, the iterates lie on a common geodesic and a sharper exact factor can be derived.*

**Remark 4.6 (Role of the golden-ratio parameter).** *The Fejér argument above is valid for any fixed relaxation parameter $\theta \in (0,1)$ after replacing $\vartheta$ by $\theta$. Thus the golden ratio is a prescribed relaxation choice rather than a parameter whose universal optimality follows from the convergence proof. The specific contribution studied here is the integration of this golden-ratio relaxation with the CAT(1) resolvent and Mosco-stability framework.*

## 5. Mosco Stability

Let $\{f_m\}$ and $f$ be proper lower semicontinuous semiconvex functions defined on the same admissible region $C$. Assume that their semiconvexity moduli and the admissible parameter $\lambda > 0$ satisfy a common resolvent condition ensuring existence and uniqueness.

**Assumption 5.1 (Mosco–resolvent stability).** *Assume*

$$f_m \xrightarrow{M} f,$$

*and that the hypotheses of the CAT(1) Mosco-resolvent convergence theorem are satisfied uniformly in $m$. In particular, whenever $y_m \to y$ strongly in $C$,*

$$J_\lambda^{f_m}(y_m) \to J_\lambda^{f}(y). \tag{5.1}$$

### 5.1. Stability of finite GRRA trajectories

For each $m$, define

$$z_n^{(m)} = J_\lambda^{f_m}(x_n^{(m)}), \qquad x_{n+1}^{(m)} = (1-\vartheta)x_n^{(m)} \oplus \vartheta z_n^{(m)}. \tag{5.2}$$

For the limit problem define

$$z_n = J_\lambda^{f}(x_n), \qquad x_{n+1} = (1-\vartheta)x_n \oplus \vartheta z_n.$$

**Theorem 5.2.** *Assume $x_0^{(m)} \to x_0$ and Assumption 5.1 holds. Then, for every fixed $n \geq 0$,*

$$x_n^{(m)} \to x_n \qquad \text{and} \qquad z_n^{(m)} \to z_n.$$

**Proof.** The assertion for $n = 0$ follows from the hypothesis. Suppose $x_n^{(m)} \to x_n$. By (5.1),

$$z_n^{(m)} = J_\lambda^{f_m}(x_n^{(m)}) \to J_\lambda^{f}(x_n) = z_n.$$

On a common admissible CAT(1) region, geodesic interpolation depends continuously on its endpoints. Hence

$$x_{n+1}^{(m)} = (1-\vartheta)x_n^{(m)} \oplus \vartheta z_n^{(m)} \to (1-\vartheta)x_n \oplus \vartheta z_n = x_{n+1}.$$

The result follows by induction.

### 5.2. Quantitative perturbation estimate

A quantitative recurrence requires more than continuity of geodesic interpolation. We therefore state the needed local Lipschitz property explicitly.

**Assumption 5.3 (Local Lipschitz stability of interpolation).** *There exists $G_C \geq 1$ such that, for all $x, x', y, y' \in C$,*

$$d\big((1-\vartheta)x \oplus \vartheta y,\ (1-\vartheta)x' \oplus \vartheta y'\big) \leq G_C \left[(1-\vartheta)d(x,x') + \vartheta d(y,y')\right]. \tag{5.3}$$

Assume additionally that

$$\sup_{x\in C} d\left(J_\lambda^{f_m} x, J_\lambda^f x\right) \leq \varepsilon_m, \qquad \varepsilon_m \to 0, \tag{5.4}$$

and that $J_\lambda^f$ is $L$-Lipschitz on $C$:

$$d(J_\lambda^f x, J_\lambda^f y) \leq L d(x,y). \tag{5.5}$$

Let

$$e_n^{(m)} := d(x_n^{(m)}, x_n).$$

Then

$$d(z_n^{(m)}, z_n) \leq \varepsilon_m + L e_n^{(m)}.$$

Using Assumption 5.3,

$$e_{n+1}^{(m)} \leq \kappa e_n^{(m)} + G_C \vartheta \varepsilon_m, \qquad \kappa := G_C\big[(1-\vartheta) + \vartheta L\big]. \tag{5.6}$$

If $\kappa < 1$, then

$$e_n^{(m)} \leq \kappa^n e_0^{(m)} + \frac{G_C \vartheta}{1-\kappa}\varepsilon_m. \tag{5.7}$$

### 5.3. Joint convergence

Suppose each $f_m$ has a unique minimizer $p_m$, $f$ has the unique minimizer $p$, and

$$p_m \to p.$$

Define

$$T_m x = (1-\vartheta)x \oplus \vartheta J_\lambda^{f_m} x.$$

Assume that there exists $\rho \in [0,1)$, independent of $m$, such that

$$d(T_m x, p_m) \leq \rho d(x, p_m), \qquad x \in C. \tag{5.8}$$

Then

$$d(x_n^{(m)}, p) \leq \rho^n d(x_0^{(m)}, p_m) + d(p_m, p). \tag{5.9}$$

Consequently, if $m = m(n) \to \infty$ and $\{d(x_0^{(m)}, p_m)\}$ is bounded, then

$$x_n^{(m(n))} \to p.$$

## 6. Numerical Example

This section illustrates the theory using a spherical quadratic minimization problem. The example is chosen so that all relevant points remain in one common admissible region of diameter below $\pi/2$.

### 6.1. Spherical quadratic problem

Let $\mathbb{S}^2 = \{x \in \mathbb{R}^3 : \|x\|_2 = 1\}$ with intrinsic distance $d(x, y) = \arccos\langle x, y\rangle$. Let $a = (0, 0, 1)$ and define the meridian $\gamma(r) = (\sin r, 0, \cos r)$. Choose $x_0 = \gamma(0.9)$. All iterates in the experiment remain on the geodesic segment joining $a$ and $x_0$. This segment is contained in a common geodesically convex spherical neighborhood whose diameter is below $\pi/2$, so the admissibility requirements are respected.

Consider

$$f(x) = \frac{\mu}{2} d^2(x, a), \qquad \mu > 0. \tag{6.1}$$

The unique minimizer is $a$. For $\tau > 0$,

$$J_\tau^f(x) = \operatorname{argmin}_y \left\{ \frac{\mu}{2} d^2(y, a) + \frac{1}{2\tau} d^2(x, y) \right\}.$$

The minimizer lies on the geodesic from $x$ to $a$, and a one-dimensional minimization gives

$$J_\tau^f(x) = (1 - s)x \oplus sa, \qquad s = \frac{\mu\tau}{1 + \mu\tau}. \tag{6.2}$$

Therefore

$$d(J_\tau^f(x), a) = \frac{1}{1 + \mu\tau} d(x, a). \tag{6.3}$$

### 6.2. Exact convergence factor of GRRA

For the fixed step $\tau$,

$$z_n = J_\tau^f(x_n), \qquad x_{n+1} = (1 - \vartheta)x_n \oplus \vartheta z_n.$$

Since $a, z_n, x_n$ lie on the same geodesic,

$$\begin{aligned} d(x_{n+1}, a) &= (1 - \vartheta) d(x_n, a) + \vartheta d(z_n, a) \\ &= \left[ (1 - \vartheta) + \frac{\vartheta}{1 + \mu\tau} \right] d(x_n, a). \end{aligned} \tag{6.4}$$

Hence,

$$d(x_n, a) = \rho_{\mathrm{GR}}^n d(x_0, a), \qquad \rho_{\mathrm{GR}} = 1 - \vartheta \frac{\mu\tau}{1 + \mu\tau}. \tag{6.5}$$

For

$$\mu = 1, \qquad \tau = 0.5, \qquad d(x_0, a) = 0.9,$$

we have

$$\vartheta = \frac{1}{\phi} \approx 0.6180339887, \qquad \rho_{\mathrm{GR}} \approx 0.7939886704.$$

Thus

$$d(x_n, a) = 0.9(0.7939886704)^n.$$

### 6.3. Mosco perturbation

Let

$$\delta_m = \frac{0.6}{m}, \qquad a_m = \gamma(\delta_m),$$

and define

$$f_m(x) = \frac{1}{2}d^2(x, a_m).$$

Because $0 \le \delta_m \le 0.6 < 0.9$, all $a_m$ lie on the same admissible geodesic segment joining $a$ and $x_0$. Moreover,

$$d(a_m, a) = \frac{0.6}{m} \to 0.$$

On the common admissible region, $f_m \to f$ uniformly. Since these spherical quadratic functions are continuous and geodesically convex on the region under consideration, the weak liminf and strong recovery conditions are satisfied, and hence

$$f_m \xrightarrow{M} f.$$

For each perturbed problem,

$$J_\tau^{f_m}(x) = \frac{1}{1+\tau}x \oplus \frac{\tau}{1+\tau}a_m,$$

and therefore

$$d(x_n^{(m)}, a_m) = \rho_{\mathrm{GR}}^n d(x_0, a_m).$$

The triangle inequality gives

$$d(x_n^{(m)}, a) \le \rho_{\mathrm{GR}}^n d(x_0, a_m) + \frac{0.6}{m}. \tag{6.6}$$

This explicitly separates the iteration error from the perturbation error.

### 6.4. Comparison with fixed-parameter Krasnosel'skiĭ–Mann-type relaxations

The classical averaging framework originating with Krasnosel'skiĭ [13] and Mann [14] motivates the fixed-parameter resolvent relaxation

$$x_{n+1}^{(\beta)} = (1-\beta)x_n^{(\beta)} \oplus \beta J_\tau^f(x_n^{(\beta)}), \qquad 0 < \beta \le 1. \tag{6.7}$$

The choices $\beta = 0.30$ and $\beta = 0.50$ are not treated as two intrinsically different algorithms; they are two fixed-relaxation baseline instances of the same averaging scheme.

For (6.1),

$$\rho_\beta = 1 - \beta\frac{\mu\tau}{1+\mu\tau}, \qquad d(x_n^{(\beta)}, a) = \rho_\beta^n d(x_0, a). \tag{6.8}$$

For $\mu = 1$ and $\tau = 0.5$,

$$\rho_{0.30} = 0.900000, \qquad \rho_{0.50} = 0.833333, \qquad \rho_{1/\phi} = 0.793989.$$

**Table 1**
Comparison of fixed-parameter Krasnosel'skiĭ-Mann-type resolvent relaxations and GRRA for $\mu = 1$, $\tau = 0.5$, and $d(x_0, a) = 0.9$.

| Method | $\beta$ | $\rho_\beta$ | $d(x_{10}, a)$ | $d(x_{20}, a)$ | $d(x_{30}, a)$ |
|---|---|---|---|---|---|
| KM(0.30) | 0.300000 | 0.900000 | $3.1381 \times 10^{-1}$ | $1.0942 \times 10^{-1}$ | $3.8152 \times 10^{-2}$ |
| KM(0.50) | 0.500000 | 0.833333 | $1.4536 \times 10^{-1}$ | $2.3476 \times 10^{-2}$ | $3.7914 \times 10^{-3}$ |
| GRRA | $1/\phi \approx 0.618034$ | 0.793989 | $8.9616 \times 10^{-2}$ | $8.9234 \times 10^{-3}$ | $8.8853 \times 10^{-4}$ |

Table 1 and Figure 1 show exact geometric decay for all three tested parameters. Among these three choices, the golden-ratio value gives the smallest contraction factor. This does *not* establish universal optimality. Indeed,

$$\rho_\beta = 1 - \beta\frac{\mu\tau}{1+\mu\tau}$$

is strictly decreasing in $\beta$ for this particular model. A larger admissible relaxation parameter would therefore yield a smaller factor. The experiment should be read as a consistency illustration rather than as evidence that $\beta = 1/\phi$ is optimal among all relaxations.

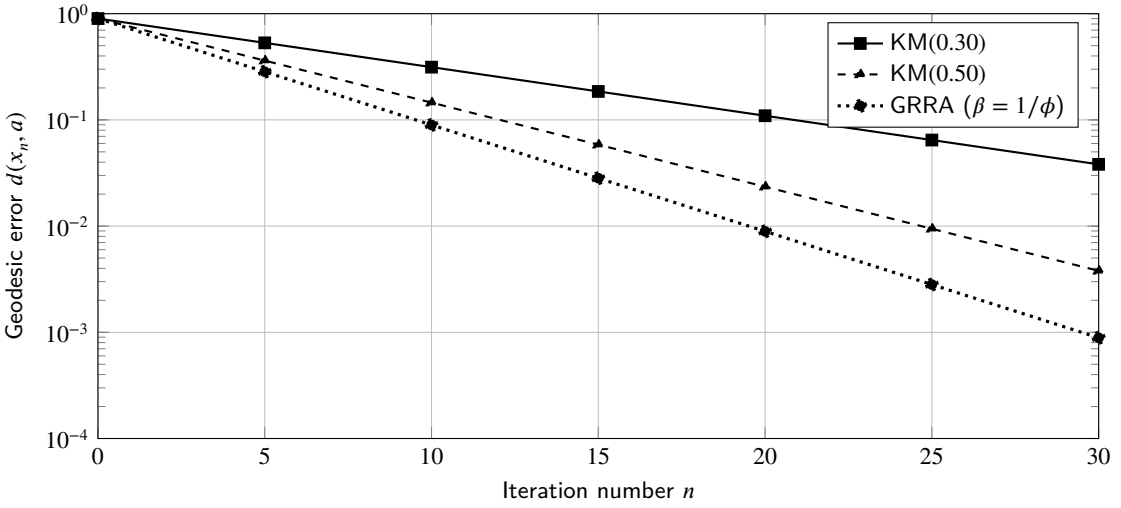



**Figure 1:** Geometric decay of the geodesic error for two fixed-parameter Krasnosel'skiĭ–Mann-type resolvent relaxations and GRRA. Among the three tested parameters, the golden-ratio choice has the smallest contraction factor.

## 7. Discussion and Conclusion

We developed an intrinsic golden-ratio relaxed resolvent method for semiconvex optimization on admissible regions of complete CAT(1) spaces. The revised analysis makes explicit the two issues that are crucial in positive curvature and semiconvex optimization.

First, the convergence proof uses a local strong-convexity inequality for the squared distance only on a common admissible region where that estimate is valid. No universal coefficient is assigned to every CAT(1) ball merely from a radius condition. Second, for semiconvex objectives that may be nonconvex, resolvent fixed points are not identified with global minimizers. Instead, global convergence is derived under explicit quasi-nonexpansiveness and $\Delta$-demiclosedness assumptions relative to the minimizer set.

Under these hypotheses, GRRA is Fejér monotone, its resolvent residual is square summable, and the sequence is asymptotically regular. With $\Delta$-demiclosedness, the iterates $\Delta$-converge to a minimizer. Under a contractive resolvent assumption, strong convergence follows with the general metric rate

$$\rho = \sqrt{1 - \vartheta(1 - q^2)}.$$

The Mosco analysis gives stability of every fixed finite GRRA trajectory. A quantitative perturbation recurrence is valid when a local Lipschitz estimate for geodesic interpolation is available.

The spherical quadratic example yields a sharper exact scalar factor because all points lie on a common geodesic. The numerical comparison shows that the golden-ratio relaxation is faster than the two smaller relaxation parameters tested, but it does not establish universal optimality of the golden-ratio choice. More broadly, the convergence proof itself extends to any fixed relaxation parameter in $(0, 1)$. Accordingly, the contribution of the present work is best understood as the integration of a prescribed golden-ratio geodesic relaxation with a curvature-sensitive resolvent and Mosco-stability framework in CAT(1) geometry.

Future work may investigate parameter selection rules that exploit curvature information adaptively, conditions under which the quasi-nonexpansiveness and $\Delta$-demiclosedness assumptions follow directly from structural hypotheses on the objective, and extensions to equilibrium and variational inequality problems in positively curved spaces.

## Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Author Contributions

Kenyi Calderón and Duangkamon Kitkuan: Conceptualization, methodology, formal analysis, writing-original draft. Anantachai Padcharoen: Conceptualization, methodology, validation, writing-review and editing, supervision. The authors have read and approved the final manuscript.

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